Decomposition of noncommutative $U(1)$ gauge potential

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Abstract

We investigate the decomposition of noncommutative gauge potential $\hat{A}_i$, and find it has inner structure, namely, $\hat{A}_i$ can be decomposed in two parts $\hat{b}_i$ and $\hat{a}_i$, here $\hat{b}_i$ satisfies gauge transformations while $\hat{a}_i$ satisfies adjoint transformations, so dose the Seiberg-Witten mapping of noncommutative $U(1)$ gauge potential. By means of Seiberg-Witten mapping, we construct a mapping of unit vector field between noncommutative space and ordinary space, and find the noncommutative $U(1)$ gauge potential and its gauge field tenser can be expressed in terms of the unit vector field. When the unit vector field has non singularity point, noncommutative gauge potential and gauge field tenser will equal to ordinary gauge potential and gauge field tenser.

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I. INTRODUCTION

The decomposition theory of the gauge potential in terms of unit vector field was first assumed by Duan and Ge.[1], then subsequently by Cho[2, 3]. In 1999, Faddeev and Niemi proposed a decomposition of the four dimensional $SU(N)$ Yang- Mills field A^a_μ[4, 5], recently, they also introduced a novel, complete field decomposition in the Yang-Mills Lagrangian, and suggested that due to fluctuations in the gauge invariant condensate ρ and the vector field $\mathbf{n}$, at short distances both Newtons constant and the cosmological constant become variable[6]. The decomposition theory of the gauge potential may provide a very powerful tool in establishing direct relationship between differential geometry and topological invariants in studding some topological properties of physical systems. This has been effectively used to study the magnetic monopole problem in $SU(2)$ gauge theory[7, 8, 9], the topological structure of the Gauss-Bonnet-Chern theorem and Morse theory[10], the topological gauge theory of dislocation and declinations in condensed matter physics[11, 12], the torsion structure on Riemann-Cartan manifold and knots[13], the topological structure of Chern-Simons vortex[14].

Recently the classical properties of noncommutative gauge theories have been investigated in many directions[15, 16]. In fact the Space-Time coordinates dose not commute was first assumed by Snyder in 1947[17], and based on the Snyder construction, C. N. Yang obtain another space-time that is compatible with the Lorentz symmetry and also it is invariant under translations, this space-time is discrete in the spatial coordinates[18]. The general noncommutative geometry was established by A. Conies, and Yang-Mills theory on a noncommutative torus has been proposed as an example[20], for noncommutative manifolds $\mathbb{R}^n$, the coordinates x^i whose commutators are c-numbers,

$$[x^i, x^j] = i\theta^{ij}, \tag{1}$$

with real θ, where θ^{ij} is antisymmetric matrix, i.e.

$$\theta^{ij} = -\theta^{ji}, \tag{2}$$

the product on noncommutative space is Moyal $\star$ product which can be defined as[19]

$$f \star g = fg + \frac{i}{2}\theta^{ij}\partial_i f \partial_j g + \mathcal{O}(\theta^2). \tag{3}$$

For noncommutative Yang-Mills theory, because of the $U(1)$ universality as noted by Gross and Nekrasov[16], it seems that it is possible to everything about the $U(N)$ gauge theory can be found in the $U(1)$ theory[22]. In 1999, Seiberg and Witten find an equivalence between ordinary gauge fields and noncommutative gauge fields, which is realized by a change of variables that can be described explicitly[21],

$$\hat{A}_i = A_i - \frac{1}{4}\theta^{kl}\{A_k, \partial_l A_i + F_{li}\} + \mathcal{O}(\theta^2), \tag{4}$$

$$\hat{\alpha} = \alpha + \frac{1}{4}\theta^{ij}\{\partial_i \alpha, A_j\} + \mathcal{O}(\theta^2), \tag{5}$$

this is called Seiberg-Witten mapping, where the product on the right hand side such as $\{A_k, \partial_l A_i\} = A_k \cdot \partial_l A_i + \partial_l A_i \cdot A_k$ are ordinary matrix products, $\hat{A}_i$ is noncommutative gauge potential, while A_i is ordinary gauge potential, $\hat{\alpha}$ is noncommutative gauge parameter, α is ordinary gauge parameter, and so on.

In this paper, we will study the decomposition of noncommutative $U(1)$ gauge potential by using Seiberg-Witten mapping. We decompose the $U(1)$ Seiberg-Witten mapping in two parts, one part satisfies gauge transformation, and the other satisfies an adjoint transformation, and construct a mapping between unit vector field in noncommutative space and in ordinary space, we discuss the inner structure of noncommutative gauge potential and find the noncommutative $U(1)$ gauge potential and its gauge field tenser can be expressed in terms of the unit vector field. When the unit vector field has non singularity point, noncommutative gauge potential and gauge field tenser will equal to ordinary gauge potential and gauge field strength.

II. DECOMPOSITION OF THE SEIBERG-WITTEN MAPPING OF $U(1)$ GAUGE POTENTIAL

The gauge transformations of noncommutative Yang-Mills theory are thus

$$\hat{A}_i \longrightarrow \hat{S} \star \hat{A}_i \star \hat{S}^{-1} + \partial_i \hat{S} \star \hat{S}^{-1}, \tag{6}$$

in which $\hat{A}_i$ is noncommutative gauge potential and

$$\hat{S} = exp(i\hat{\alpha})_{\star} \tag{7}$$

is noncommutative gauge group, using Eq.(3) we can get

$$f \star (g + h) = f \star g + f \star h \tag{8}$$

Suppose the noncommutative gauge potential $\hat{A}_i$ can be written in two parts

$$\hat{A}_i = \hat{b}_i + \hat{a}_i, \tag{9}$$

sub Eq.(9) to Eq.(6), and using Eq.(8), we can get

$$\hat{A}_i \longrightarrow \hat{S} \star \hat{b}_i \star \hat{S}^{-1} + \hat{S} \star \hat{a}_i \star \hat{S}^{-1} + \partial_i \hat{S} \star \hat{S}^{-1}, \tag{10}$$

so can suppose $\hat{b}_i$ satisfies gauge transforation and $\hat{a}_i$ satisfies an adjoint transformation

$$\hat{b}_i \longrightarrow \hat{S} \star \hat{b}_i \star \hat{S}^{-1} + \partial_i \hat{S} \star \hat{S}^{-1}, \tag{11}$$

$$\hat{a}_i \rightarrow \hat{S} \star \hat{a}_i \star \hat{S}^{-1}. \tag{12}$$

the ordinary gauge potential can be written as[1]

$$A_i = b_i + a_i, \tag{13}$$

$$b_i \longrightarrow S b_i S^{-1} + \partial_i S S^{-1}, \tag{14}$$

$$a_i \rightarrow S a_i S^{-1}. \tag{15}$$

now we want find a mapping between $\hat{b}_i$ and b_i, and a mapping between $\hat{a}_i$and a_i, this mapping should satisfies Seiberg-Witten mapping, that is, for $U(1)$ group, those mapping must satisfies

$$\hat{a}_i + \hat{b}_i = a_i + b_i + \frac{1}{2}\theta^{kl}(a_l + b_l)(2\partial_k(a_i + b_i) - \partial_i(a_k + b_k)), \tag{16}$$

like Seiberg-Witten mapping, we can suppose

$$\hat{a}_i = a_i + \frac{1}{2}\theta^{kl} f(A_i), \tag{17}$$

$$\hat{b}_i = b_i + \frac{1}{2}\theta^{kl} g(A_i), \tag{18}$$

where $f(A_i)$ and $g(A_i)$ is the function of A_i, according to the principle of Seiberg-Witten mapping, the mapping of $\hat{a}_i(A_i)$ should satisfies

$$\hat{a}_i{}'(A_i) = \hat{a}_i(A_i^{'}), \tag{19}$$

in which

$$\hat{a}_i{}' = \hat{S} \star \hat{a}_i \star \hat{S}^{-1}, \tag{20}$$

and

$$A_i^{'} = SA_iS^{-1} + \partial_i SS^{-1}. \tag{21}$$

Equation (19) is solved by

$$\hat{a}_i = a_i + \theta^{kl} A_l \partial_k a_i - \frac{1}{2}\theta^{kl} a_l \partial_i a_k, \tag{22}$$

$$\hat{b}_i = b_i + \frac{1}{2}\theta^{kl}[A_l(2\partial_k b_i - \partial_i b_k) - b_l \partial_i a_k]. \tag{23}$$

here we assume $b = 0 \Leftrightarrow \hat{b} = 0$, so we decompose the $U(1)$ Seiberg-Witten mapping in two parts, one part satisfies gauge transformation (14), and the other satisfies an adjoint transformation (15).

III. SEIBERG-WITTEN MAPPING OF UNIT VECTOR FIELD

In order to decompose the noncommutative $U(1)$ gauge potential in terms of unit vector field, we will construct an unit vector field in commutative space, then establish a mapping between unit vector field in noncommutative space and unit vector field in ordinary space. Complex scalar field in noncommutative space is thus

$$\hat{\phi} = \hat{\phi}_1 + i\hat{\phi}_2, \tag{24}$$

and complex scalar field in ordinary space is

$$\phi = \phi_1 + i\phi_2, \tag{25}$$

using Seiberg-Witten mapping of complex scalar field[23],

$$\hat{\phi} = \phi + \frac{1}{2}\theta^{ij} A_j \partial_i \phi, \tag{26}$$

we can get

$$\hat{\phi}_1 = \phi_1 + \frac{1}{2}\theta^{ij} A_j \partial_i \phi_1, \tag{27}$$

$$\hat{\phi}_2 = \phi_2 + \frac{1}{2}\theta^{ij} A_j \partial_i \phi_2, \tag{28}$$

it is obvious their anti mapping is

$$\phi_1 = \hat{\phi}_1 - \frac{1}{2}\theta^{ij} \hat{A}_j \partial_i \hat{\phi}_1, \tag{29}$$

$$\phi_2 = \hat{\phi}_2 - \frac{1}{2}\theta^{ij} \hat{A}_j \partial_i \hat{\phi}_2, \tag{30}$$

we define a unit vector in noncommutative space as

$$\hat{n^a} = \frac{\hat{\phi}^a}{\sqrt{\hat{\rho}}}, \tag{31}$$

where

$$\hat{\rho} = \hat{\phi}^a \hat{\phi}^a \tag{32}$$

it naturally has the constraint

$$\hat{n^a}\hat{n^a} = 1. \tag{33}$$

It is easy to prove that

$$\hat{n^a} = n^a + \frac{1}{2}\theta^{ij} A_j \partial_i n^a, \tag{34}$$

and its anti mapping

$$n^a = \hat{n^a} - \frac{1}{2}\theta^{ij} \hat{A}_j \partial_i \hat{n^a}, \tag{35}$$

IV. DECOMPOSITION OF NONCOMMUTATIVE $U(1)$ GAUGE POTENTIAL

The decomposition of ordinary $U(1)$ gauge potential is[1, 13],

$$A_i = \varepsilon_{ab} n^a \partial_i n^b - \varepsilon_{ab} n^a D_i n^b, \tag{36}$$

sub Eq.(35) into Eq.(36), we can obtain

$$\begin{aligned} A_i = {} & \varepsilon_{ab}\hat{n^a}\partial_i\hat{n^b} - \varepsilon_{ab}\hat{n^a} D_i \hat{n^b} - \frac{1}{2}\theta^{kl}\varepsilon_{ab}[\hat{n^a}\partial_i \hat{A}_l \partial_k \hat{n^b} \\ & + \hat{A}_l \partial_k \hat{n^a} \partial_i \hat{n^b} - \hat{n^a} D_i(\hat{A}_l \partial_k \hat{n^b}) - \hat{A}_l \partial_k \hat{n^a} D_i \hat{n^b}], \end{aligned} \tag{37}$$

and the noncommutative gauge potential can be decomposed in follows

$$\begin{aligned}\hat{A}_i = \varepsilon_{ab}\hat{n^a}\partial_i\hat{n^b} - \varepsilon_{ab}\hat{n^a}D_i\hat{n^b} - \frac{1}{2}\theta^{kl}\varepsilon_{ab}[\hat{n^a}\partial_i\hat{A}_l\partial_k\hat{n^b} \\ +\hat{A}_l\partial_k\hat{n^a}\partial_i\hat{n^b} - \hat{n^a}D_i(\hat{A}_l\partial_k\hat{n^b}) - \hat{A}_l\partial_k\hat{n^a}D_i\hat{n^b}] + \\ \frac{1}{2}\theta^{kl}(\varepsilon_{ab}\hat{n^a}\partial_l\hat{n^b} - \varepsilon_{ab}\hat{n^a}D_l\hat{n^b})\cdot[2\partial_k(\varepsilon_{cd}\hat{n^c}\partial_i\hat{n^d} \\ -\varepsilon_{cd}\hat{n^c}D_i\hat{n^d}) - \partial_i(\varepsilon_{cd}\hat{n^c}\partial_k\hat{n^d} - \varepsilon_{cd}\hat{n^c}D_k\hat{n^d})],\end{aligned} \tag{38}$$

when

$$Dn = 0, \tag{39}$$

using Eq.(36) we can get

$$A_{i//} = \varepsilon_{ab}n^a\partial_i n^b, \tag{40}$$

sub it to Seiberg-Witten mapping

$$\hat{A}_{i//} = \varepsilon_{ab}n^a\partial_i n^b + \frac{3}{2}\theta^{kl}\varepsilon_{ab}\varepsilon_{cd}n^a\partial_l n^b\partial_i n^d\partial_k n^c, \tag{41}$$

because $a, b = 1, 2; c, d = 1, 2$,

$$\hat{A}_{i//} = A_{i//}[1 + \frac{3}{4}\theta^{kl}\varepsilon_{ab}\partial_k n^a\partial_l n^b], \tag{42}$$

using

$$\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i - i\hat{A}_i \star \hat{A}_j + i\hat{A}_j \star \hat{A}_i, \tag{43}$$

we can obtain

$$\hat{F}_{ij//} = [1 + \frac{3}{4}\theta^{kl}\varepsilon_{ab}\partial_k n^a\partial_l n^b]F_{ij//} + \theta^{kl}\partial_k A_{i//}\partial_l A_{j//}, \tag{44}$$

where $F_{ij//}$ is ordinary gauge field strength,

$$\begin{aligned}F_{ij//} &= \partial_i A_{j//} - \partial_j A_{i//} \\ &= 2\varepsilon_{ab}\partial_i n^a\partial_j n^b,\end{aligned} \tag{45}$$

so Eq.(44) can be rewritten as

$$\hat{F}_{ij//} = [1 + \theta^{kl}\varepsilon_{ab}\partial_k n^a\partial_l n^b]F_{ij//}, \tag{46}$$

according to ϕ-mapping theory[10], we can obtain

$$\varepsilon_{ab}\partial_k n^a\partial_l n^b = \pi\delta(\overrightarrow{\phi})\varepsilon_{ab}\partial_k\phi^a\partial_l\phi^b, \tag{47}$$

and

$$\varepsilon_{ab}\partial_k\phi^a\partial_l\phi^b = \varepsilon_{kl\lambda_1\cdots\lambda_{m-2}} J^{\lambda_1\cdots\lambda_{m-2}}\left(\frac{\overrightarrow{\phi}}{\overrightarrow{x}}\right), \tag{48}$$

where $\overrightarrow{x}$ is vector in ordinary space, m is the dimension of space. Sub Eq.(47) and Eq.(48) to Eq.(42) we can get

$$\hat{A}_{i//} = A_{i//}[1 + \frac{3}{4}\pi\theta^{kl}\delta(\overrightarrow{\phi})\varepsilon_{kl\lambda_1\cdots\lambda_{m-2}} J^{\lambda_1\cdots\lambda_{m-2}}\left(\frac{\overrightarrow{\phi}}{\overrightarrow{x}}\right)], \tag{49}$$

while

$$\hat{F}_{ij//} = F_{ij//}[1 + \pi\theta^{kl}\delta(\overrightarrow{\phi})\varepsilon_{kl\lambda_1\cdots\lambda_{m-2}} J^{\lambda_1\cdots\lambda_{m-2}}\left(\frac{\overrightarrow{\phi}}{\overrightarrow{x}}\right)], \tag{50}$$

in which $k, l, \lambda_1, \cdots \lambda_{m-2} = 1, 2, \cdots m$, from this equation we can see, when there is non singularity point, noncommutative gauge potential and gauge field tenser will equal to ordinary gauge potential and gauge field strength.

V. CONCLUSIONS

The gauge field in noncommutative space is different from ordinary gauge field, this two fields can connected by the mapping which supposed by Seiberg and Witten. The general decomposition theory of gauge potential in ordinary space has been established by way of geometric algebra, in which the gauge potential can be decomposed in terms of unit vector field. In this paper, we prove the noncommutative $U(1)$ gauge potential also has inner structure, and constructed the decomposition of noncommutative $U(1)$ gauge potential in terms of unit vector field by means of Seiberg-Witten mapping, unlike the decomposition of ordinary gauge potential, there are two kinds of unit vector field, the one is the unit vector field in ordinary space, the other is the unit vector field in noncommutative space, and we construct a mapping between unit vector field in noncommutative space and in ordinary space. We also find the Seiberg-Witten mapping can be decompose in two parts, one part satisfies gauge transformation, and the other satisfies an adjoint transformation. Finally, we calculate gauge potential and the gauge field tenser and find if the unit vector field in ordinary space has non singularity point, noncommutative gauge potential and gauge field tenser will equal to ordinary gauge potential and gauge field strength.

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